

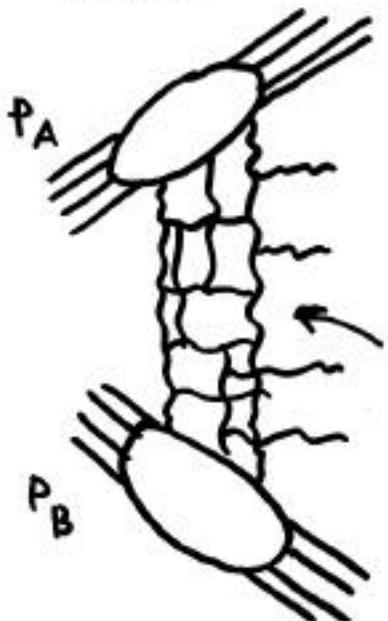
# Factorization and high-energy effective action

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Plan:

1. Factorization for high-energy scattering
2. Effective action
3. Semiclassical approach:  
scattering of two shock waves
4.  $S_{\text{eff}}$  in the leading log approximation

$s \gg p_A^2, p_B^2$  - Regge limit



$$p_A = p_1 + \frac{p_A^2}{s} p_2$$

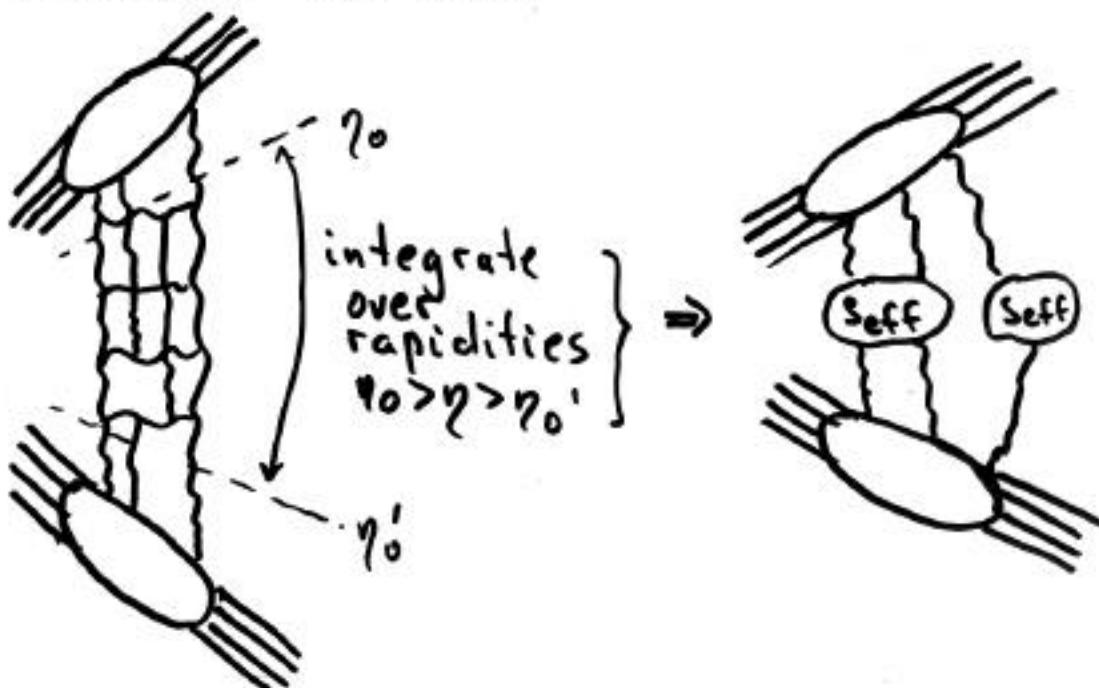
$$p_B = p_2 + \frac{p_B^2}{s} p_1$$

$\alpha = \alpha p_1 + \beta p_2 + \gamma$  -  
- Sudakov variables

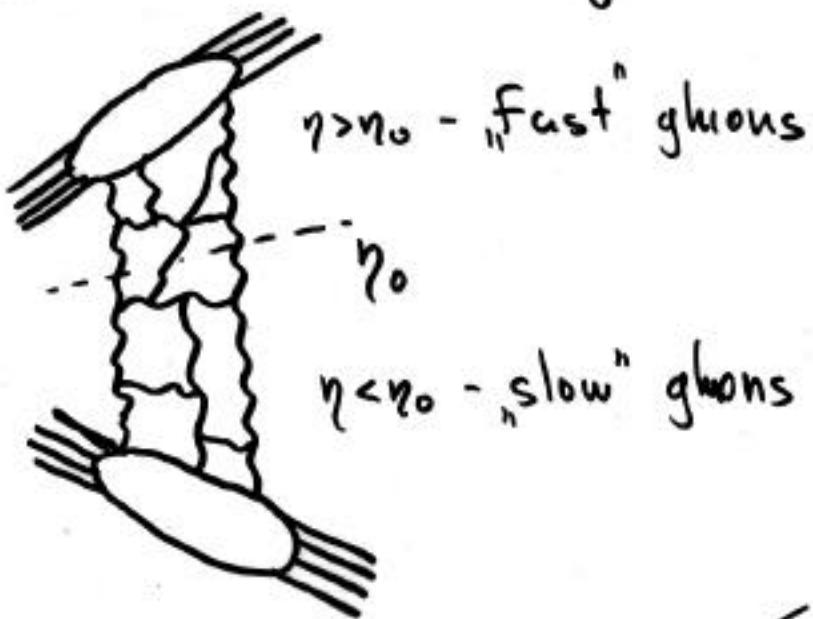
Suppose  $\alpha(k_\perp) \ll 1 \Rightarrow$   
(McLerran)

→ we can use pQCD  
and/or semiclassical QCD

Effective action:



# Factorization for high-energy scattering



$$A_\mu \rightarrow A_{\mu}^{\text{fast}} + A_{\mu}^{\text{slow}} \Rightarrow S(A) \xrightarrow{\text{QCD action}} S(A) + S(B) + S(C)$$

? ✓

Key observation:

$$\begin{array}{c} \text{fast gluon} \\ \{ \quad \} \\ \text{slow gluons} \end{array} \Rightarrow \begin{array}{c} \cdot \{ \cdot \} \cdot \{ \cdot \} \cdot \\ \text{eikonal factor} \\ (\text{Wilson lines}) \end{array}$$

$$U(x_1) = \text{Pexp} i \int_{\substack{\text{line } \parallel p_i}} A_\mu dx_\mu$$

Factorization formula

$$\int dA e^{iS(A)} =$$

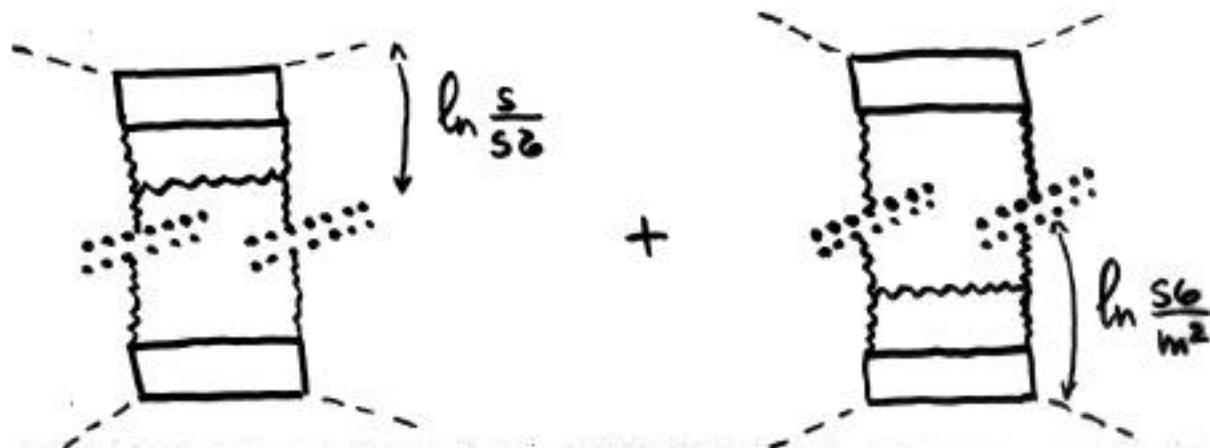
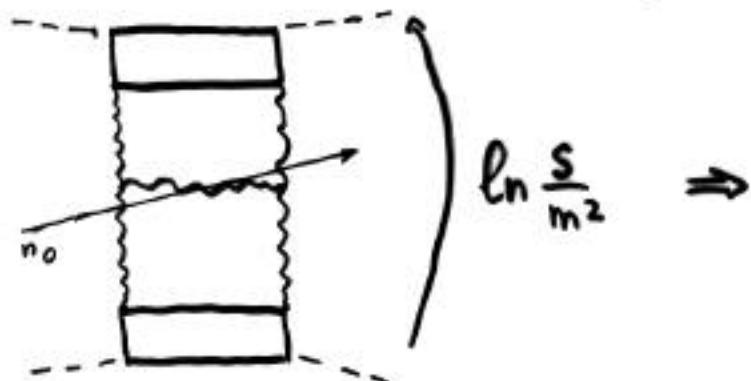
$$= \int dA \int dB \exp[iS(A) + iS(B) + i\{d^2 x_i U_i V_i + \text{O}(g^3)\}]$$

$$U_i = U + i\partial_i U, \quad V_i = V + i\partial_i V$$

$$U = P \exp[i g \int_{-\infty}^{\infty} A_\mu dx_\mu] \quad V = P \exp[i g \int_{-\infty}^{\infty} B_\mu dx_\mu]$$

infinite straight-line ordered gauge links  
(Wilson-line operators). Line  $\uparrow \uparrow n = 2p_1 + 2p_2$

Cartoon (for  $\gamma^* \gamma^*$  scattering):  $(\eta_0 = \ln \frac{s}{\Lambda^2})$



effective action for high-energy scattering

definition of eff. action:

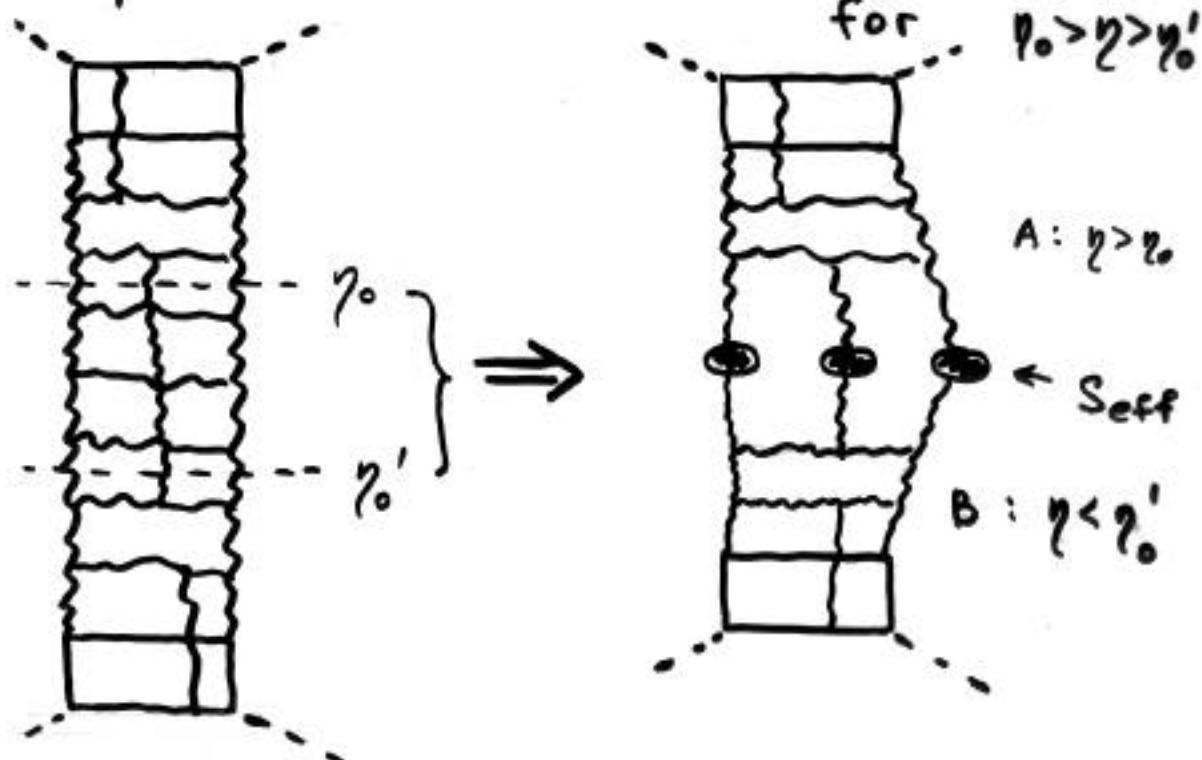
$$\langle 0 | T \{ j(p_A) j(p'_A) j(p_B) j(p'_B) \} | 0 \rangle =$$

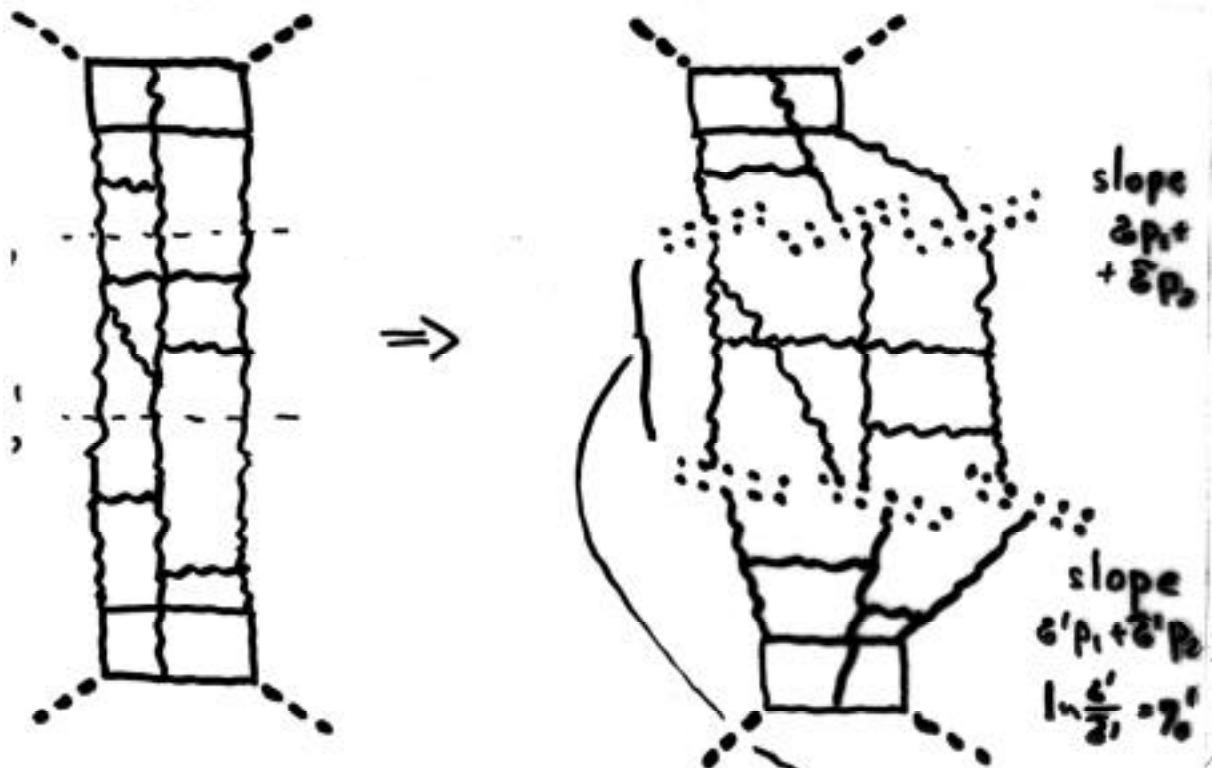
$$\int dA j_A(p_A) j_A(p'_A) e^{iS(A)} . \quad A: \eta > \eta_0$$

$$\int dB j_B(p_B) j_B(p'_B) e^{iS(B)} . \quad B: \eta < \eta'_0$$

$$\exp i S_{\text{eff}}(A, B)$$

$S_{\text{eff}}$  accounts  
for  $\eta_0 > \eta > \eta'_0$





$$\begin{aligned}
 & \oint_A j(p_A) j(p_A') j(p_B) j(p_B') e^{iS} = \\
 & \left( \oint_A j_A j_{A'} e^{iS(A)} \right) \left( \oint_B j_B j_{B'} e^{iS(B)} \right) \exp :S_{\text{eff}}^*(A, B) : \\
 & \quad \eta > \eta_0 \qquad \qquad \eta < \eta_0' \qquad \qquad \eta'_* > \eta > \eta_0
 \end{aligned}$$

normal definition of  $S_{\text{eff}}$

$$\geq S_{\text{eff}}(V, Y; \ln \frac{\omega_1}{\omega_2}) =$$

$$\int dA e^{i \oint_{\partial A} (A)} \exp \left( i \int d^2 x_1 V_i^a(x_1) V_i^a(x_1) + i \int d^2 x_1 W_i^a(x_1) Y_i^a(x_1) \right)$$

$$\begin{aligned} V_i &\equiv V^+ i \partial_i V && \left\{ \text{external sources} \right. \\ Y_i &\equiv Y^+ i \partial_i Y && \left. \right\} \end{aligned}$$

$$V_i(x_1) = V^+ i \partial_i V \quad V(x_1) = [-\omega n_1 + x_+, \omega n_1 + x_-] \equiv [-\omega n_1, \omega n_1]_x$$

$$W_i(x_1) = W^+ i \partial_i W \quad W(x_1) = [-\omega n_2 + x_+, \omega n_2 + x_-] \equiv [-\omega n_2, \omega n_2]_x$$

$$[x, y] = P \exp \left( \int du (x-y)_\mu A_\mu (u x + (1-u)y) \right)$$

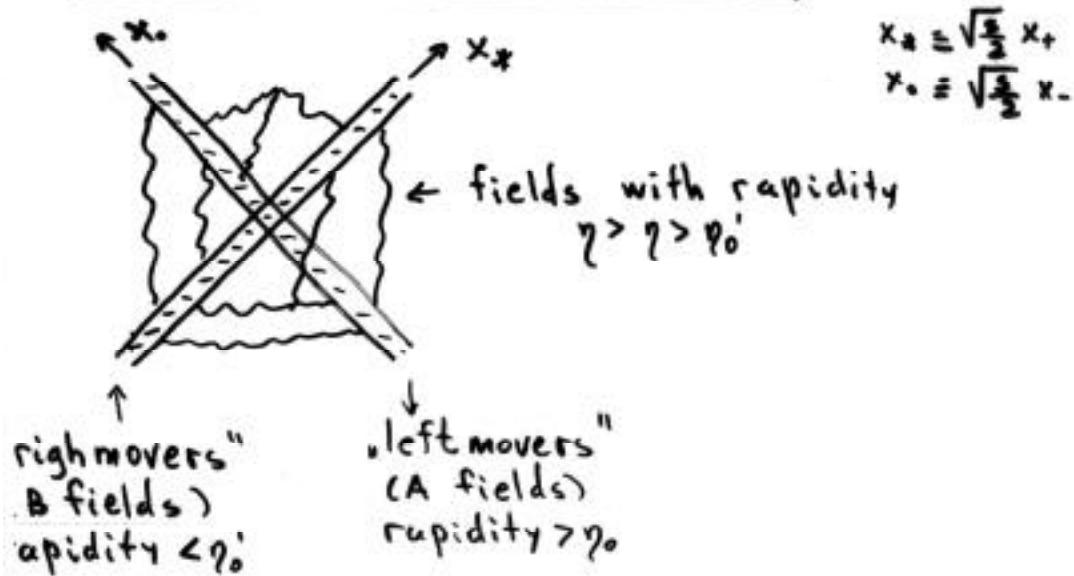
vectors  $n_1$  and  $n_2$  correspond to rapidities

$\eta_0$  and  $\eta'_0$ . In LLA,  $n_1 \leftrightarrow p_1$  and  $n_2 \leftrightarrow p_2$

$\Rightarrow$  we have functional integral with <sup>two</sup> sources

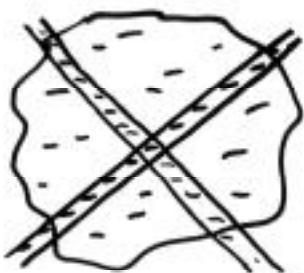
lying on the planes  $x_+ = 0$  and  $x_- = 0 \Rightarrow$

collision of two shock waves



## Semiclassical approach

$$\frac{\delta S_{\text{eff}}}{\delta A} \Big|_{A=\bar{A}} = 0$$



$\bar{A}$  = classical field generated by the collision of two shock waves

Semiclassical approach is relevant when  $\alpha_s \ll 1$  but the fields ( $\Leftrightarrow$  the sources) are strong ( $V_i \sim Y_i \sim \frac{1}{g} \Leftrightarrow V, Y \sim 1$ )

Classical eqns (explicit form):

$$D_\mu F_{\mu 2} = 0$$

$$D_\mu F_{\mu 0} = \delta(x_0) [x \cdot p_2, -\omega p_2]_{x_2} (\delta_{ij} - i[V_i]) [-\omega p_2, x \cdot p_2]_{x_2}$$

$$D_\mu F_{\mu 1} = \delta(x_1) [x \cdot p_1, -\omega p_1]_{x_2} (\delta_{ij} - i[U_i]) [-\omega p_1, x \cdot p_1]_{x_2}$$

non-local "charges"

Possible way to solve classical eqs:  
 Take a trial configuration = sum of shock waves

$$A_i^{(0)} = Y_i \Theta(x_*) + V_i \Theta(x_*) ; A_\perp = A_{\perp 0} = 0$$

and improve it order by order in pert. theory

$$(D^2 \delta_{\mu\nu} - 2i G_{\mu\nu})^{(0)} \delta A_\nu^{(1)} = \frac{\delta S_{\text{eff}}}{8A} \Big|_{A=A^{(0)}}$$

Parameter of the expansion is

$$\sim [Y_i, V_j] g^2 \ln \frac{g}{\epsilon} \rightarrow \text{sort of leading log approximation}$$

In the first nontrivial order

$$F_{ik}^{(1)} = g \int dz_\perp \frac{1}{-x_{\parallel i}^2 + (x-z)_\perp^2} [Y_k, V^k](z_\perp)$$

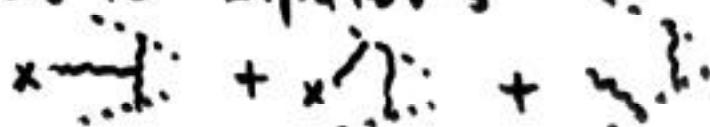
$$F_{ik}^{(1)} \approx g \int dz_\perp \frac{1}{-x_{\parallel i}^2 + (x-z)_\perp^2} ([Y_i, V_k] - i\omega k)$$

$$F_{ik}^{(1)} = \left( \frac{1}{x_{\parallel i} + i\omega} \right) g \int dz_\perp \frac{(x-z)_\perp^k}{-x_{\parallel i}^2 + (x-z)_\perp^2} L_{ik}$$

$$x_{\parallel i}^2 \sim x_* x.$$

$$L_{ik} \equiv \delta_{ik} [Y_j, V_j] + [Y_i, V_k] - [Y_k, V_i]$$

This corresponds to Lipatov's eff. vertex



The corresponding effective action is

$$\frac{i\alpha_s}{4\pi} \ln \frac{z}{g_1} \int dx_\perp dy_\perp \left\{ L_{ik}(x_\perp) (x_\perp \frac{1}{p_\perp^2} |y_\perp|) L_{ik}(y_\perp) + \dots \right\}$$

$$(x_\perp \frac{1}{p_\perp^2} |y_\perp|) = \int d^2 p_\perp e^{-ip(x-y)_\perp} = -\frac{1}{\pi} \ln(x-y)^2$$

Structure:

$$S_{\text{eff}}^{(1)} = \frac{i\alpha_s}{4\pi} \ln \frac{z}{g_1} \left[ x_\perp \{y_\perp + \dots\} \frac{z}{z-y} \{y + \{x_\perp \frac{z}{z-y}\}\} \right]$$

$$x_\perp y_\perp = (x_\perp \frac{1}{p_\perp^2} |y_\perp|) \text{ or } (x_\perp \frac{p_k}{p_\perp^2} |y_\perp|)$$

$$\frac{z}{z-y} = V_i \text{ (or } \partial_i V) \quad \{y\} = \Psi_i \text{ (or } \partial_i \Psi)$$

To compare with perturbative results, we must add gluon reggeization. In this approach, gluon reggeization comes from the first quantum correction (due to non-local source it is logarithmic in  $\beta/g_1$ ).

$$S_{\text{reg}}^{(1)} \sim \frac{\alpha_s}{2\pi^2} \ln \frac{z}{g_1} \int d^2 x_\perp d^2 y_\perp (Y^i(x_\perp) V^i(y_\perp) - Y^i(x_\perp) V_i(y_\perp)) \frac{1}{(x-y)^2}$$

$$\frac{1}{(x-y)^2} = (x_\perp \ln p_\perp^2 |y_\perp|)$$

$\alpha a \equiv \text{Tr in adjoint representation}$

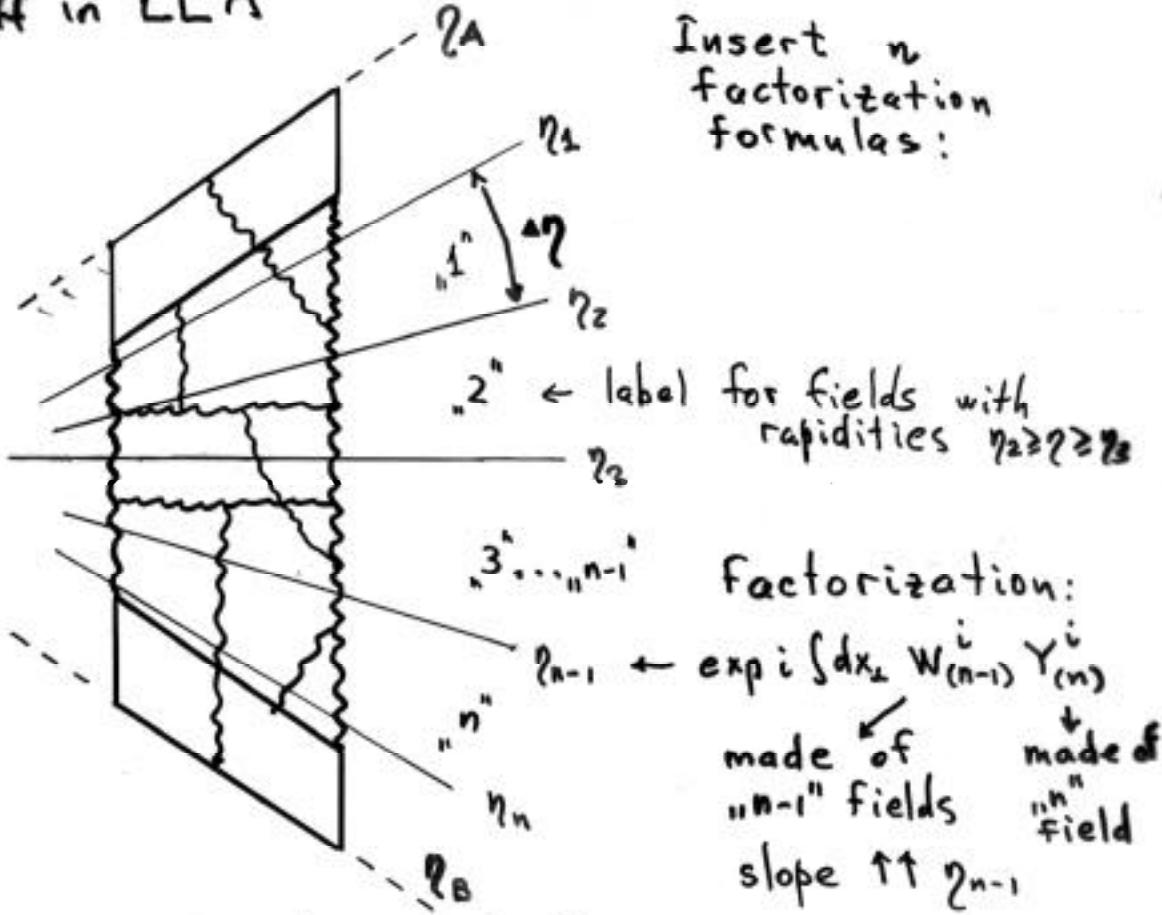
from the gluon loop:

$$\text{Structure: } S_{\text{reg}}^{(1)} = \text{loop} \quad \int d^2 k_\perp \frac{p^2}{k^2(p-k)^2} \simeq \ln p_\perp^2$$

Thus, in the first nontrivial order

$$S_{\text{eff}} = i \int d^2 x_\perp V_i Y^i + S_{\text{eff}}^{(1)} + S_{\text{reg}}^{(1)}$$

eff in LLA

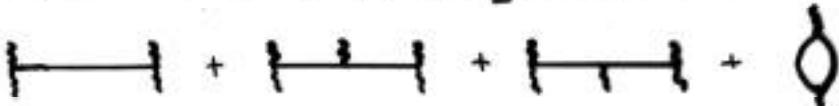


one rung of the "ladder":

$$S_{eff}(V, Y_{(2)}) = S_{eff}(A_{(1)}) e^{iS(A_{(1)})} e^{iS_{dx_1}\{V^k U_{(1)}^k + W_{(1)}^k Y_{(2)}^k\}}$$

$$\exp[iS_{dx_1} V^k Y_{(2)}^k + \Delta \eta K(V, Y_{(2)})]$$

$$(V, Y) = S_{dx_1} [V_i, Y^i] \frac{1}{\partial_x^2} [V_j, Y^j] + \dots =$$



one-loop kernel of the evolution

two rungs:

$$\geq iS_{\text{eff}}(V, Y_{(3)}) = \int dA_1 e^{iS(A_1)} \int dA_2 e^{iS(A_2)} e^{i \int dk_L (V^k U_1^k + W_1^k Y_2^k + W_2^k Y_3^k)}$$

Trick:

$$\geq i \int W^k Y_k = \det(D_k \tilde{\nabla}^k) \int d\Omega d\Lambda e^{i \int W^k \Omega_k + i \int \Lambda^k Y_k - i \int \Lambda^k \Omega_k}$$

can be  
neglected  
in LLA

$$g \Lambda_k \equiv \Lambda^+ i \partial_k \Lambda$$
$$g \Omega_k \equiv \Omega^+ i \partial_k \Omega$$
$$\nabla_i \equiv \partial_i - g [W_i]$$
$$\tilde{\nabla}_i \equiv \partial_i - g [Y_i]$$

$$\geq e^{iS_{\text{eff}}(V, Y_3)} =$$
$$= \int d\Omega d\Lambda \int dA_1 e^{iS(A_1)} e^{i \int V^k U_1^k + i \int W_1^k \Omega^k} e^{-i \int \Lambda^k \Omega^k}$$
$$\cdot \int dA_2 e^{iS(A_2)} e^{i \int \Lambda^k Y_2^k + i \int W_2^k Y_3^k} =$$
$$= \int d\Omega d\Lambda e^{i \int V^k \Omega^k + \alpha \eta K(V, \Omega)} e^{-i \int \Lambda^k \Omega^k} e^{i \int \Lambda^k Y_3^k + K(\dots)}$$
$$= \int d\Omega d\Lambda \exp i \int V^k \Omega^k + \alpha \eta K(V, \Omega) - i \int \Lambda^k \Omega^k + i \int \Lambda^k Y_3^k + \alpha \eta K(\Lambda, Y_3)$$

n rungs:

$$e^{iS_{\text{eff}}(V, Y)} =$$

$$= \int \prod_{m=1}^n D\Omega_m D\Lambda_m \exp \left\{ i \int V^k \Omega_1^k + \Delta \eta K(V, \Omega_1) - i \int \Omega_1^k \Lambda_1^k + \right.$$

$$+ \Delta \eta K(\Lambda_1, \Omega_2) + i \int \Lambda_1 \Omega_2 + \Delta \eta K(\Lambda_2, \Omega_3) - i \int \Omega_2 \Lambda_2^k + i \int \Lambda_2^k \Omega_3^k +$$

$$\left. + \dots i \int \Lambda_n^k Y_k \right\}$$

$n \rightarrow \infty$ :

$$e^{iS_{\text{eff}}(V, Y)} = \int D\Lambda(\eta, x_1) D\Omega(\eta, x_1) \exp \left[ i \int V^k \Omega_1^k + \int_{\eta_A}^{\eta_B} d\eta \left\{ -i \int dx_1 \Lambda_1^k(x_1, \eta) \frac{d}{dx_1} \Omega_1^k(x_1, \eta) \right. \right. \quad (*)$$

$$\left. \left. + K(\Lambda(x_1, \eta), \Omega(x_1, \eta)) \right\} \right].$$

boundary conditions

non-local Hamiltonian

Instead of integration over gluons, we have an integration over collective "string" degrees of freedom

$$\Lambda_i(x_1, \eta) \sim P \exp i \int \lambda_\mu (x_1 + \lambda n) n_\mu d\lambda \quad n \mapsto e_\eta$$

Classical equations for (\*)

$$\frac{\partial}{\partial \eta} \Omega = \frac{i}{2\pi} t^a \left( - \left( \Omega^+ \frac{\partial \Omega^-}{\partial x_1} \Omega \Omega^- \frac{1}{\partial x_1^2} \right)^{ab} L^b - \left( \frac{\partial \Omega^+}{\partial x_1} \Omega^+ \frac{\partial \Omega^-}{\partial x_1} \Omega^- \right)^a L^b \right)$$

$$L = [\Omega_k, \Lambda_k] \quad L_{kl} = [\Omega_k, \Lambda_l] - \text{cyclic}$$

$$\Lambda^+ \frac{\partial}{\partial \eta} \Lambda = \text{similar}$$

Perturbative expansion  $\Rightarrow$  field theory for reggeized gluons

$$\Phi(x_\perp, \eta) = e^{ig\psi(x_\perp, \eta)}$$

$$\psi \equiv \psi^a t_a$$

$$\Lambda(x_\perp, \eta) = e^{ig\pi(x_\perp, \eta)}$$

$$\pi \equiv \pi^a t_a$$

$$e^{iS_{\text{eff}}(v, \eta)} =$$

$$t = g^2 \eta$$

$$= \int d\theta \psi(x_\perp, t) \partial_T \pi(x_\perp, t) e^{i \int dx_\perp v \partial^2 \psi(t_\perp)}.$$

$\mu = \text{IR cutoff}$

$$\cdot \exp \int_{t_B}^{t_A} dt \int dx_\perp \left[ -i\pi \left( \frac{\partial}{\partial t} + \frac{N_c}{4\pi^2} \ln \frac{\partial_\perp^2}{\mu^2} \right) \partial_\perp^2 \psi + [\bar{\partial}\psi, \bar{\partial}\pi] \frac{1}{\partial_\perp^2 + \mu^2} [\partial\psi, \bar{\partial}\pi] \right]$$

gluon reggeization      BFKL kernel

Lipatov's Hamiltonian for quantum mechanics of reggeized gluons

$$\begin{aligned} &+ g [\bar{\partial}\psi, \bar{\partial}\pi] \psi \frac{1}{\partial_\perp^2 + \mu^2} [\partial\psi, \bar{\partial}\pi] + g \partial_i \pi \ln \frac{\partial_\perp^2}{\mu^2} [\psi, \partial_i \psi] + \\ &+ g [\bar{\partial}\psi, \bar{\partial}\pi] \frac{\partial}{\partial_\perp^2 + \mu^2} \psi \frac{\partial}{\partial_\perp^2 + \mu^2} [\partial\psi, \bar{\partial}\pi] + \pi \rightarrow \psi + \dots \end{aligned}$$

production (and annihilation) of reggeized gluons

Propagator of the reggeized gluon

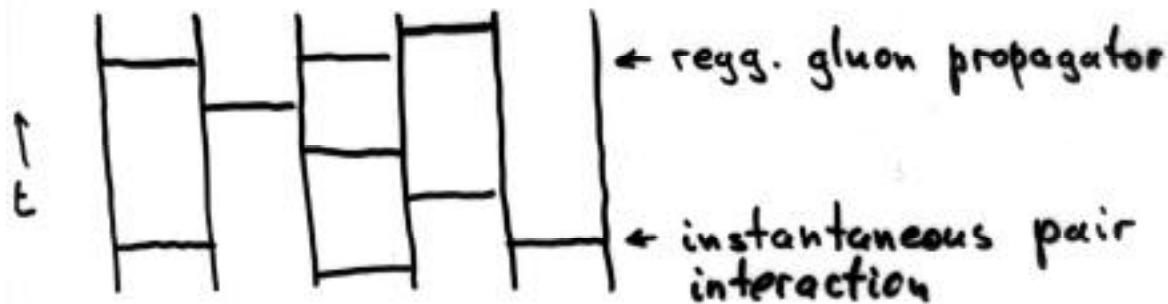
$$\overline{\psi(x_\perp, \eta)} \pi(y_\perp, t') = \Theta(t - t') \int dp_\perp e^{-ip(x-y)_\perp} \frac{1}{p_\perp^2} e^{(\eta - \eta') \propto (p_\perp^2)}$$

✓  
trajectory  
of the reggeized  
gluon ( $\sim \ln p^2$ )

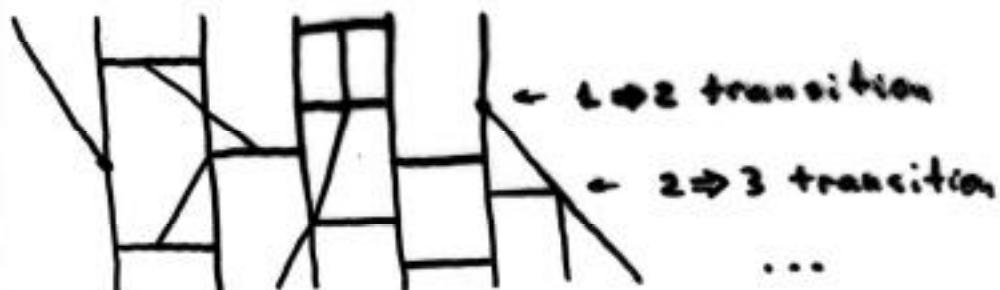
Vertices are non-local

# Feynman diagrams for the field theory of reggeized gluons

1. Lipatov's Hamiltonian - no regg. gluon production  
 $\Leftrightarrow$  spin 0 Heisenberg model (Lipatov, Faddeev, Korchemsky)



2. Real thing



At large  $\gamma_A - \gamma_B$  ( $\Leftrightarrow$  large  $T = t_h - t_n$ )  
 the result should look like

$$\langle \Psi | \Psi \rangle \propto e^{iS(\varphi, \pi)} \xrightarrow[T \rightarrow 0]{\quad} \Psi_0(v) e^{-T \omega_0} \Psi_0(y)$$

wave functional  
of the "vacuum"  
 $(\equiv$  pomeron  $)$

vacuum  
energy  
 $(\equiv$  position  
of the pomeron  $)$

## Conclusions

1. Factorization formula  $\Rightarrow$  rigorous definition of  $S_{\text{eff}}$  for a given interval in rapidity
2. Semiclassical approach to  $S_{\text{eff}} \Leftrightarrow$  scattering of two shock waves in QCD
3. Functional integral for  $S_{\text{eff}}$  in terms of Wilson-line variables effectively summarizes all known information about high-energy scattering (BFKL, three-pomeron vertex,  $\chi\chi\chi$  spin-0 Heisenberg model for reggeized gluons - everything except NLO<sub>BFKL</sub>)

## Outlook:

1. Take into account particle production in the final state ( $S_{\text{eff}}$  for inclusive cross sections)
2. Numerical calculations of Wilson-line functional integral ( $w_0$  from lattice)?
3.  $S_{\text{eff}}$  in NLO